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New lower solution bounds of the continuous algebraic Riccati matrix equation

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Abstract

In optimal and robust control problems, the so-called continuous algebraic Riccati equation (CARE) plays an important part. By utilizing some matrix inequalities and linear algebraic techniques, new lower matrix bounds for the solution of the CARE are derived. Following the derived bounds, iterative algorithms are then developed to obtain sharper solution estimates. In comparison to existing results, the obtained bounds are less restrictive. Finally, we give numerical examples to demonstrate the effectiveness of our results.

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1. Introduction and preliminaries

Consider the continuous algebraic Riccati equation (CARE):

$$PA + A^T P - P B B^T P = -Q, \quad (1.1)$$

where $A \in \mathbb{R}^{n \times n}$ is a constant matrix, $(^T)$ denotes the transpose, $Q \in \mathbb{R}^{n \times n}$ is a symmetric positive semi-definite non-zero matrix, and the matrix P is the unique symmetric positive semi-definite solution of the CARE (1.1). To guarantee the existence of such a unique stabilizing solution, it shall be assumed that (A, B) is a controllable pair and the pair $(A, Q^{1/2})$ is an observable pair.

The continuous Riccati equation is usually encountered in optimal control problems for continuous-time control systems. In practice, analytical solution of this equation is often

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computationally difficult and time-consuming, particularly as the dimensions of the system matrices increase. Solution bounds can be used to reduce the computational efforts required to solve the equation and give a rough estimate for its actual solution. Further to this, solution bounds can also be applied to deal with practical applications involving the solution of the CARE. As such, a number of works have been presented over the past three decades for deriving solution bounds of this equation, many of which are summarized in [8,24]. Types of solution bound include extremal eigenvalue [2,3,5,15,17,18], eigenvalue summation [3], trace [4,6,7,16,19], eigenvalue product (which are derivable from matrix bounds), determinant [3,22], norm [4,20] and matrix bounds [9–14,21,23]. Of these bounds, the matrix bounds are the most general and desirable, since they can infer all other types of bounds mentioned. However, viewing the literature, it seems that all proposed lower matrix bounds [9–13,21] for the CARE either have to assume the matrix Q is nonsingular, or that Q is singular and A is in the range space of Q . These are very restrictive assumptions, because such assumptions are not common in control problems involving the solution of the CARE. The only available lower matrix bound for the CARE that can deal with this case is that proposed in [9]. However, when Q is singular, $\lambda_n(Q) = 0$, and the result of [9] yields the trivial lower bound $P \geq 0$. By using a method similar to Lee's method [14], this note develops three new lower matrix bounds to remove this assumption and yield nontrivial lower matrix bounds for the CARE. It is not necessary to assume that Q is nonsingular for these results. Following each lower bound, iterative algorithms are also developed to obtain sharper bounds. Finally, we give numerical examples to demonstrate the effectiveness of the present results.

The following notations are adopted in this paper: \Re denotes the real number field. $X > (\geq) Y$ means matrix $X - Y$ is positive (semi-)definite; $\lambda_i(X)$ and $\sigma_i(X)$ denote, respectively, the i th eigenvalue and i th singular value of matrix X for $i = 1, 2, \dots, n$, and are arranged in the non-increasing order $\lambda_1(X) \geq \lambda_2(X) \geq \dots \geq \lambda_n(X)$ and $\sigma_1(X) \geq \sigma_2(X) \geq \dots \geq \sigma_n(X)$. $\mu(X)$ denotes the matrix measure of a matrix $X \in \Re^{n \times n}$, defined by $\mu(X) \equiv \frac{1}{2}\lambda_1(X + X^T)$, and $\text{Re}(\lambda(X))$ is the real part of an eigenvalue of X . Furthermore, I denotes the identity matrix with suitable dimensions.

Before developing the main results, we shall review the following useful results.

Lemma 1.1 [1]. *For any symmetric matrices $X, Y \in \Re^{n \times n}$ and $1 \leq i, j \leq n$, the following inequality holds:*

$$\lambda_{i+j-n}(X + Y) \geq \lambda_j(X) + \lambda_i(Y), \quad i + j \geq n + 1 \quad \text{and} \quad 1 \leq i, j \leq n.$$

Lemma 1.2 [1]. *For any symmetric matrix X , the following inequality holds:*

$$\lambda_n(X)I \leq X \leq \lambda_1(X)I.$$

Lemma 1.3 [1]. *For any matrices $X, Y \in \Re^{n \times n}$ and any matrix $A \in \Re^{n \times m}$, if $X \geq Y$ then*

$$A^T X A \geq A^T Y A.$$

Lemma 1.4 [1]. *For any positive semi-definite matrices $X, Y \in \Re^{n \times n}$ such that $0 \leq X \leq Y$, the following inequality holds:*

$$X^2 \leq (XY^2X)^{1/2}.$$

Lemma 1.5 [23]. *The positive semi-definite solution P of the CARE (1.1) has the following upper bound on its maximal eigenvalue:*

$$\lambda_1(P) \leq \lambda_1(D^T D) \frac{\lambda_1[(Q + K^T K)D^T D]}{\lambda_n(MD^T D)} \equiv \eta, \quad (1.2)$$

where K is any matrix stabilizing $A + BK$ (i.e., $\text{Re}(\lambda_i(A + BK)) < 0$ for all i) and the nonsingular matrix D and positive definite matrix M are chosen to yield the LMI

$$(A + BK)^T D^T D + D^T D(A + BK) \leq -M.$$

This eigenvalue upper bound is always calculated if there exists a unique, non-negative definite solution of the CARE (1.1).

2. Main results

In this section, we shall develop the main results. First, a lower matrix bound for the CARE is derived, followed by the development of an iterative algorithm to obtain tighter solution estimates. Then, a second lower matrix bound shall be derived, also accompanied by an iterative algorithm to obtain sharper solution estimates. Finally, a third lower matrix bound for the solution of the CARE is developed, along with an iterative algorithm to obtain more precise lower matrix bounds.

Theorem 2.1. Define

$$V \equiv A - \alpha I - I, \quad (2.1)$$

where α is a positive constant. Let P be the positive semi-definite solution of the CARE (1.1). If

$$A + A^T < \lambda_1(BB^T)\eta I, \quad (2.2)$$

where η is defined by (1.2), then P has the lower bound

$$P \geq V^{-T}(\varphi_1[(V + I)^T(V + I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)I] + Q)V^{-1} \equiv P_{l1}, \quad (2.3)$$

where the positive constant α is chosen so that

$$A + A^T < 2\alpha I \quad (2.4)$$

and

$$2\alpha + 1 \geq \lambda_1(BB^T)\eta \quad (2.5)$$

are satisfied, and the non-negative constant φ_1 is defined by

$$\varphi_1 \equiv \frac{\lambda_n[V^{-T}QV^{-1}]}{1 - \lambda_n\{V^{-T}[(V + I)^T(V + I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)I]V^{-1}\}}.$$

Proof. The CARE (1.1) can be rewritten as

$$P(A - \alpha I) + (A - \alpha I)^T P + 2\alpha P + Q = PBB^T P, \quad (2.6)$$

where α is a positive constant.

Using the definition of V from (2.1), we can define the following matrix identity:

$$V^T P V = (V + I)^T P (V + I) - (A - \alpha I)^T P - P(A - \alpha I) + P. \quad (2.7)$$

Via (2.7), (2.6) can be rewritten as

$$(V + I)^T P (V + I) + (2\alpha + 1)P + Q = PBB^T P + V^T P V. \quad (2.8)$$

From Lemma 1.2 we have $BB^T \leq \lambda_1(BB^T)I$. Then, applying Lemma 1.3 to the term PBB^TP gives $PBB^TP \leq \lambda_1(BB^T)P^2$.

Since $0 \leq P \leq \lambda_1(P)I$ from Lemma 1.2, we have from Lemma 1.4 that

$$P^2 \leq [P\lambda_1^2(P)IP]^{1/2} = \lambda_1(P)P. \quad (2.9)$$

Substituting (1.2) into (2.9) and combining this with the previous results for PBB^TP gives

$$PBB^TP \leq \lambda_1(BB^T)\eta P. \quad (2.10)$$

Substituting (2.10) into (2.8) gives

$$\begin{aligned} (V+I)^TP(V+I) + (2\alpha+1)P + Q &\leq \lambda_1(BB^T)\eta P + V^TPV \\ \Rightarrow V^TPV &\geq (V+I)^TP(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)P + Q. \end{aligned} \quad (2.11)$$

Since $\operatorname{Re} \lambda(A - \alpha I) \leq \mu(A - \alpha I) \equiv \frac{1}{2}\lambda_1(A + A^T - 2\alpha I)$, one can see that choosing α to meet the condition (2.4) ensures the nonsingularity of V . Furthermore, if α is chosen to meet condition (2.5), then the term $(2\alpha+1-\lambda_1(BB^T)\eta)P$ is non-negative definite, from which the main result follows. Therefore, we have from (2.11) that

$$P \geq V^{-T}[(V+I)^TP(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)P + Q]V^{-1}. \quad (2.12)$$

Application of Lemma 1.2 to (2.12) gives

$$P \geq V^{-T}[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]\lambda_n(P) + Q)V^{-1}. \quad (2.13)$$

Introducing Lemma 1.1 to (2.13) gives

$$\begin{aligned} \lambda_n(P) &\geq \lambda_n\{V^{-T}[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]\lambda_n(P) + Q\}V^{-1} \\ &\geq \lambda_n\{V^{-T}[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]V^{-1}\}\lambda_n(P) \\ &\quad + \lambda_n[V^{-T}QV^{-1}]. \end{aligned} \quad (2.14)$$

Using the well-known fact that $\lambda_i(XY) = \lambda_i(YX)$ for any $X, Y \in \mathbb{R}^{n \times n}$ and $i = 1, 2, \dots, n$, we have from (2.14) that

$$\begin{aligned} \lambda_n\{V^{-T}[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]V^{-1}\} \\ = \lambda_n\{[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I](V^TV)^{-1}\} \\ = \lambda_n\{[(A-\alpha I)^T(A-\alpha I) + (2\alpha+1-\lambda_1(BB^T)\eta)I] \\ \times [(A-\alpha I-I)^T(A-\alpha I-I)]^{-1}\}. \end{aligned}$$

Let $M \equiv (A - \alpha I)^T(A - \alpha I)$. Then (2.14) can be rewritten as

$$\begin{aligned} \lambda_n(P) &\geq \lambda_n\{V^{-T}[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]\lambda_n(P) + Q\}V^{-1} \\ &= \lambda_n\{[M + (2\alpha+1-\lambda_1(BB^T)\eta)I][M - (A-\alpha I)^T - (A-\alpha I) + I]^{-1}\}\lambda_n(P) \\ &\quad + \lambda_n[V^{-T}QV^{-1}]. \end{aligned}$$

Since $M \geq 0$, it is seen that if condition (2.2) is met, then

$$\lambda_n\{V^{-T}[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]V^{-1}\} < 1.$$

Then, from (2.14) we have that

$$\lambda_n(P) \geq \frac{\lambda_n[V^{-T}QV^{-1}]}{1 - \lambda_n\{V^{-T}[(V+I)^T(V+I) + (2\alpha+1-\lambda_1(BB^T)\eta)I]V^{-1}\}} \equiv \varphi_1. \quad (2.15)$$

Substituting (2.15) into (2.13) leads to the lower bound (2.3). This completes the proof of the theorem. \square

Remark 2.1. Since I is a positive definite matrix of full rank, there will always exist a positive constant α such that condition (2.4) is fulfilled, as well as condition (2.5). Also, since η is an upper bound for $\lambda_1(P)$, there will always exist a value of η such that the condition (2.2) is met. Hence, the lower bound P_{l1} is always calculated if the CARE solution exists. Furthermore, the above bound obtained in Theorem 2.1 has taken into account the case when Q is positive definite. When Q is positive semi-definite, $\lambda_n(Q) = 0$. As such, $V^{-T}QV^{-1}$ is also positive semi-definite, which implies that $\lambda_n(V^{-T}QV^{-1}) = 0$. In this case, $\varphi_1 = 0$ and the lower bound (2.3) becomes $P \geq V^{-T}QV^{-1}$. In fact, one can readily deduce this lower bound from (2.3), since $P_{l1} \geq V^{-T}QV^{-1}$.

Following Theorem 2.1, we can propose the following iterative algorithm to obtain tighter solution estimates for the CARE (1.1). The algorithm shall be developed for two cases of the matrix Q , namely when $Q > 0$ and $Q \geq 0$.

Algorithm 1

Step 1. Set $M_0 \equiv P_{l1}$, where P_{l1} is defined by (2.3).

Step 2. Calculate

$$M_k = V^{-T}[(V + I)^T M_{k-1} (V + I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)M_{k-1} + Q]V^{-1} \\ k = 1, 2, \dots \quad (2.16)$$

Then M_k are also lower bounds for the solution of the CARE (1.1).

Proof of Case 1. $Q > 0$: Set $k = 1$ in (2.16) to get

$$M_1 = V^{-T}[(V + I)^T M_0 (V + I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)M_0 + Q]V^{-1}. \quad (2.17)$$

Applying Lemma 1.2 to (2.17) gives

$$M_1 \geq V^{-T}[(V + I)^T (V + I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)I]\lambda_n(M_0) + Q)V^{-1}. \quad (2.18)$$

Since $M_0 = P_{l1}$, we have

$$\lambda_n(M_0) \geq \lambda_n\{V^{-T}[\varphi_1[(V + I)^T (V + I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)I] + Q]V^{-1}\} \\ \geq \varphi_1 \lambda_n\{V^{-T}[(V + I)^T (V + I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)I]V^{-1}\} + \lambda_n[V^{-T}QV^{-1}] \\ = \varphi_1 \left\{ 1 - \frac{\lambda_n[V^{-T}QV^{-1}]}{\varphi_1} \right\} + \lambda_n[V^{-T}QV^{-1}] = \varphi_1, \quad (2.19)$$

where Lemma 1.1 and (2.15) have been employed. Substituting (2.19) into (2.18) leads to

$$M_1 \geq V^{-T}[(V + I)^T (V + I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)I]\varphi_1 + Q)V^{-1} = M_0.$$

Now assume $M_{k-1} \geq M_{k-2}$. Then

$$M_k = V^{-T}[(V + I)^T M_{k-1} (V + I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)M_{k-1} + Q]V^{-1} \\ \geq V^{-T}[(V + I)^T M_{k-2} (V + I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)M_{k-2} + Q]V^{-1} = M_{k-1}.$$

By mathematical induction, it can be concluded that $M_k \geq M_{k-1} \geq \dots \geq M_1 \geq M_0$. By the above proof and by definition of (2.12), one can see that the matrices M_k are indeed lower

bounds of the solution of the CARE (1.1). This completes the proof of the algorithm for the case $Q > 0$. \square

Proof of Case 2. $Q \geq 0$: Firstly, note that, for this case, $M_0 = V^{-T}QV^{-1}$. Set $k = 1$ in (2.16) to get (2.17). Applying Lemma 1.2 to (2.17) leads to (2.18). Since $\lambda_n(M_0) = 0$ for this case, (2.18) becomes

$$M_1 \geq V^{-T}QV^{-1} = M_0.$$

Now assume $M_{k-1} \geq M_{k-2}$. Then

$$\begin{aligned} M_k &= V^{-T}[(V+I)^T M_{k-1}(V+I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)M_{k-1} + Q]V^{-1} \\ &\geq V^{-T}[(V+I)^T M_{k-2}(V+I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)M_{k-2} + Q]V^{-1} = M_{k-1}. \end{aligned}$$

By mathematical induction, it can be concluded that $M_k \geq M_{k-1} \geq \dots \geq M_1 \geq M_0$. As with the first case, one can see that the matrices M_k are lower solution bounds of the CARE. This completes the proof of the algorithm for the case $Q \geq 0$. \square

We now obtain a different lower matrix bound as follows.

Theorem 2.2. *If the condition (2.2) is fulfilled, then the solution P of the CARE (1.1) satisfies*

$$P \geq V^{-T}(\varphi_1[(V+2I)^T(V+2I) + (4\alpha - 2\lambda_1(BB^T)\eta)I] + 2Q)V^{-1} \equiv P_{l2}, \quad (2.20)$$

where the positive constant α is chosen so as to satisfy the condition (2.4) and the condition

$$2\alpha > \lambda_1(BB^T)\eta, \quad (2.21)$$

where the non-negative constant φ_1 is defined by (2.16).

Proof. Using the definition of V from (2.1), (2.6) can be rewritten as

$$\begin{aligned} P(V+I) + (V+I)^T P + 2\alpha P + Q &= PBB^T P \\ \Rightarrow PV + V^T P + 2P + 2\alpha P + Q &= PBB^T P. \end{aligned} \quad (2.22)$$

Multiplying both sides of (2.22) by 2 and then adding $V^T P V$ to both sides of (2.24) gives

$$V^T P V + 2PV + 2V^T P + 4P + 4\alpha P + 2Q = V^T P V + 2PBB^T P. \quad (2.23)$$

Using the matrix identity

$$(V+2I)^T P(V+2I) = V^T P V + 2PV + 2V^T P + 4P,$$

(2.23) becomes

$$\begin{aligned} (V+2I)^T P(V+2I) + 4\alpha P + 2Q &= V^T P V + 2PBB^T P \\ &\leq 2\lambda_1(BB^T)\eta P + V^T P V, \end{aligned} \quad (2.24)$$

(2.24) becomes

$$V^T P V \geq (V+2I)^T P(V+2I) + (4\alpha - 2\lambda_1(BB^T)\eta)P + 2Q. \quad (2.25)$$

From (2.25), we get

$$P \geq V^{-T}[(V+2I)^T P(V+2I) + (4\alpha - 2\lambda_1(BB^T)\eta)P + 2Q]V^{-1}. \quad (2.26)$$

Application of Lemma 1.2 to (2.26) gives

$$P \geq V^{-T}[(V + 2I)^T(V + 2I) + (4\alpha - 2\lambda_1(BB^T)\eta)]\lambda_n(P) + 2Q)V^{-1}. \quad (2.27)$$

Using Lemma 1.1, we have from (2.27) that

$$\begin{aligned} \lambda_n(P) &\geq \lambda_n\{V^{-T}[(V + 2I)^T(V + 2I) + (4\alpha - 2\lambda_1(BB^T)\eta)]\lambda_n(P) + 2Q)V^{-1}\} \\ &\geq \lambda_n\{V^{-T}[(V + 2I)^T(V + 2I) + (4\alpha - 2\lambda_1(BB^T)\eta)I]V^{-1}\}\lambda_n(P) \\ &\quad + 2\lambda_n[V^{-T}QV^{-1}]. \end{aligned} \quad (2.28)$$

Following along the same lines as in Theorem 2.1, it can be seen that if condition (2.2) is met, then

$$\lambda_n\{V^{-T}[(V + 2I)^T(V + 2I) + (4\alpha - 2\lambda_1(BB^T)\eta)I]V^{-1}\} < 1.$$

From (2.28), we can then obtain

$$\begin{aligned} \lambda_n(P) &\geq \frac{2\lambda_n[V^{-T}QV^{-1}]}{1 - \lambda_n\{V^{-T}[(V + 2I)^T(V + 2I) + (4\alpha - 2\lambda_1(BB^T)\eta)I]V^{-1}\}} \\ &= \frac{\lambda_n[V^{-T}QV^{-1}]}{1 - \lambda_n\{V^{-T}[(V + I)^T(V + I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)I]V^{-1}\}} \equiv \varphi_1. \end{aligned} \quad (2.29)$$

Substituting (2.29) into (2.27) results in the lower bound (2.20). This finishes the proof of the theorem. \square

Remark 2.2. In fact, the above bound obtained in Theorem 2.2 has taken into account the case when Q is positive definite. When Q is positive semi-definite, $\lambda_n(Q) = 0$. As such, $V^{-T}QV^{-1}$ is also positive semi-definite, which implies that $\lambda_n(V^{-T}QV^{-1}) = 0$. In this case, $\varphi_1 = 0$ and the lower bound (2.20) becomes $P \geq 2V^{-T}QV^{-1} = 2P_{l1}$. Therefore, for the case when Q is singular, P_{l2} is always tighter than P_{l1} . As in Remark 2.1, it is easily seen from (2.20) that $P_{l2} \geq 2V^{-T}QV^{-1}$.

Having derived the lower bound of Theorem 2.2, we can propose the following iterative algorithm to obtain sharper solution estimates for the CARE (1.1).

Algorithm 2

Step 1. Set $\overline{M}_0 \equiv P_{l2}$, where P_{l2} is defined by (2.20).

Step 2. Calculate

$$\begin{aligned} \overline{M}_k &= V^{-T}[(V + 2I)^T\overline{M}_{k-1}(V + 2I) + (4\alpha - 2\lambda_1(BB^T)\eta)\overline{M}_{k-1} + 2Q]V^{-1}, \\ k &= 1, 2, \dots \end{aligned}$$

Then \overline{M}_k are also lower solution bounds of the CARE (1.1).

Proof. The proof of this algorithm is similar to that of Algorithm 1, and is therefore omitted. \square

Next, we shall derive a third lower matrix bound for the solution of the CARE (1.1).

Theorem 2.3. *Define*

$$W \equiv A - \beta I, \quad (2.30)$$

where β is a positive constant. Let P be the positive semi-definite solution of the CARE (1.1). If the condition (2.2) is fulfilled, then P has the lower bound

$$P \geq W^{-T}(\varphi_2[A^T A + (\beta^2 - \beta\lambda_1(BB^T)\eta)I] + \beta Q)W^{-1} \equiv P_{l3}, \quad (2.31)$$

where the constant β is chosen such that

$$A + A^T < 2\beta I, \quad (2.32)$$

$$\beta > \lambda_1(BB^T)\eta, \quad (2.33)$$

and the non-negative constant φ_2 is defined by

$$\varphi_2 \equiv \frac{\beta\lambda_n[W^{-T}QW^{-1}]}{1 - \lambda_n\{W^{-T}[A^T A + (\beta^2 - \beta\lambda_1(BB^T)\eta)I]W^{-1}\}}.$$

Proof. Using the definition of W from (2.30), we have the following matrix identity:

$$W^T P W = A^T P A - \beta(A^T P + P A) + \beta^2 P. \quad (2.34)$$

Substituting the CARE (1.1) into (2.34) gives

$$W^T P W = A^T P A - \beta P B B^T P + \beta Q + \beta^2 P. \quad (2.35)$$

From the proofs of Theorems 2.1 and 2.2, we have that $P B B^T P \leq \lambda_1(BB^T)\eta P$, where η is defined by (1.2). With this in mind, (2.35) implies

$$W^T P W \geq A^T P A + (\beta^2 - \beta\lambda_1(BB^T)\eta)P + \beta Q. \quad (2.36)$$

Along the lines of the proof of Theorem 2.1, satisfaction of (2.32) means that W is nonsingular, and (2.36) then gives

$$P \geq W^{-T}[A^T P A + (\beta^2 - \beta\lambda_1(BB^T)\eta)P + \beta Q]W^{-1}. \quad (2.37)$$

Application of Lemma 1.2 to (2.37) gives

$$P \geq W^{-T}[(A^T A + (\beta^2 - \beta\lambda_1(BB^T)\eta)I)\lambda_n(P) + \beta Q]W^{-1}. \quad (2.38)$$

Applying Lemma 1.1 to (2.38) gives

$$\begin{aligned} \lambda_n(P) &\geq \lambda_n\{W^{-T}[(A^T A + (\beta^2 - \beta\lambda_1(BB^T)\eta)I)]W^{-1}\}\lambda_n(P) \\ &\quad + \beta\lambda_n[W^{-T}QW^{-1}]. \end{aligned} \quad (2.39)$$

Along the same lines as in Theorem 2.1, it can be seen that if the conditions (2.2), (2.32) and (2.33) are satisfied, then

$$\lambda_n\{W^{-T}[(A^T A + (\beta^2 - \beta\lambda_1(BB^T)\eta)I)]W^{-1}\} < 1.$$

From (2.39), we can then obtain

$$\lambda_n(P) \geq \frac{\beta\lambda_n[W^{-T}QW^{-1}]}{1 - \lambda_n\{W^{-T}[A^T A + (\beta^2 - \beta\lambda_1(BB^T)\eta)I]W^{-1}\}} \equiv \varphi_2. \quad (2.40)$$

Substituting (2.40) into (2.38) results in the bound (2.31). This completes the proof of the theorem. \square

Remark 2.3. In fact, the above bound in Theorem 2.3 has taken into account the case when Q is positive definite. As before, when Q is positive semi-definite, the bound (2.31) becomes $P \geq \beta W^{-T} Q W^{-1}$.

Following the development of Theorem 2.3, the following iterative algorithm can be proposed to obtain more precise lower matrix bounds for the solution of the CARE (1.1).

Algorithm 3

Step 1. Set $N_0 \equiv P_{l3}$, where P_{l3} is defined by (2.31).

Step 2. Calculate

$$N_k = W^{-T}[A^T N_{k-1} A + (\beta^2 - \beta \lambda_1(BB^T)\eta)N_{k-1} + \beta Q]W^{-1}.$$

Then N_k are also lower bounds for the solution of the CARE (1.1).

Proof. The proof of this algorithm parallels that of Algorithms 1 and 2, and is hence omitted. \square

Remark 2.4. From (2.3) and (2.22), it can be seen that

$$\begin{aligned} P_{l2} &= V^{-T}(\varphi_1[(V+I)^T(V+I) + (2\alpha + 1 - \lambda_1(BB^T)\eta)I] + Q \\ &\quad + \varphi_1[(V+I) + (V+I)^T + (2\alpha - \lambda_1(BB^T)\eta)I] + Q)V^{-1} \\ &= P_{l1} + V^{-T}(\varphi_1[A + A^T - \lambda_1(BB^T)\eta I] + Q)V^{-1}. \end{aligned}$$

As such, if $\varphi_1[A + A^T - \lambda_1(BB^T)\eta I] + Q \geq 0$ then P_{l2} is tighter than P_{l1} , whereas if $\varphi_1[A + A^T - \lambda_1(BB^T)\eta I] + Q \leq 0$ then P_{l1} is tighter than P_{l2} . We find that the tightness between the bound P_{l3} and P_{l1} and P_{l2} cannot be compared mathematically.

Remark 2.5. Recently, the following lower matrix bound for the CARE (1.1) has been proposed in [12]:

$$P \geq G^{-1}[G(Q - A^T R_1 A)G]^{1/2}G^{-1} \equiv P_{l4}, \quad (2.41)$$

where the positive definite matrix R_1 is chosen such that $Q > A^T R_1 A$, and the positive definite matrix G is defined by $G = (BB^T + R_1^{-1})^{1/2}$. In [12] it was shown that, with appropriate choices of R_1 , the lower bound (2.41) is tighter and more general than existing lower matrix bounds proposed in [9–11,13,21] and the corresponding eigenvalue bounds are also sharper than most previous bounds. In [12], some choices of the matrix R_1 were listed to simplify the calculation of (2.41). Some of these choices are re-listed in the table in the second numerical example that follows. We earlier noted that nearly all existing lower matrix bounds have to assume that either Q is nonsingular or Q is singular and the matrix A is in the range space of Q . These assumptions are very conservative. Under the satisfaction of the conditions for the bounds, our bounds can always work for the case of Q being singular and nonsingular. Therefore, this note improves the assumption. Also, we do not find a mathematical method to compare the tightness between existing lower matrix bounds and those bounds presented here. However, they can supplement each other.

Remark 2.6. An iterative technique for solving the CARE (1.1) was proposed in [25]. We shall state this technique as follows: Choose a positive (semi-)definite matrix \widehat{P}_0 such that $A - BB^T \widehat{P}_0$ is a stable matrix. Also, let \widehat{P}_k be the solution of the following Lyapunov-type matrix equation:

$$\widehat{P}_k(A - BB^T \widehat{P}_{k-1}) + (A - BB^T \widehat{P}_{k-1})^T \widehat{P}_k = -(Q + \widehat{P}_{k-1} BB^T \widehat{P}_{k-1}), \quad k = 1, 2, \dots$$

Then, $\lim_{k \rightarrow \infty} \widehat{P}_k = P$, where P is the unique positive semi-definite solution of the CARE (1.1). If the matrices $A - BB^T P_{l1}$, $A - BB^T P_{l2}$ or $A - BB^T P_{l3}$ are stable, then we can choose the proposed lower bounds P_{l1} , P_{l2} or P_{l3} as the initial matrix \widehat{P}_0 and solve the CARE (1.1) by the above iterative algorithm.

3. Numerical examples

In this section, we give two numerical examples to demonstrate the effectiveness of the derived bounds. Our first example will be for the case when Q is singular. Our second example will be for the case when Q is nonsingular. Comparisons will be made with existing results when possible.

3.1. Example 1: Q is singular

Consider the CARE (1.1) with

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

For these system matrices, the unique positive definite solution of the CARE (1.1) is

$$P = \begin{bmatrix} 0.4142 & 0.4142 \\ 0.4142 & 4.4142 \end{bmatrix}.$$

Since the matrix Q is singular and A is not in the range space of Q , the lower matrix bounds proposed in [10–13] cannot work for this case. The lower matrix bound of [9] yields the trivial bound $P \geq 0$. However, the bounds of Theorems 2.1 and 2.2 can work, and give sharper results than [9]. With $\eta = 5.6754$ and $\alpha = 3$, the lower matrix bounds for the solution P of the CARE (1.1) are found by Theorems 2.1 and 2.2, respectively, to be

$$P_{l1} = \begin{bmatrix} 0.0400 & 0.1200 \\ 0.1200 & 0.3600 \end{bmatrix} \quad \text{and} \quad P_{l2} = \begin{bmatrix} 0.0800 & 0.2400 \\ 0.2400 & 0.7200 \end{bmatrix}.$$

With $\beta = 6$, the lower matrix bound P_{l3} is found by Theorem 2.3 to be

$$P_{l3} = \begin{bmatrix} 0.1224 & 0.2448 \\ 0.2448 & 0.4898 \end{bmatrix}.$$

Using two iterations of Algorithm 2, we can obtain the following tighter lower bounds for the solution of the CARE (1.1):

$$\overline{M}_1 = \begin{bmatrix} 0.1109 & 0.2470 \\ 0.2470 & 0.8562 \end{bmatrix},$$

$$\overline{M}_2 = \begin{bmatrix} 0.1228 & 0.2442 \\ 0.2442 & 0.8801 \end{bmatrix}.$$

It can be seen that as more iterations of the algorithm are carried out, the bounds become tighter.

3.2. Example 2: Q is nonsingular [3, Example 1]

Consider the CARE (1.1) with

$$A = \begin{bmatrix} 0.5 & 0 \\ 1 & -2.5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

For these system matrices, the unique positive definite solution of the CARE (1.1) is

$$P = \begin{bmatrix} 0.6989 & 0.1228 \\ 0.1228 & 0.5879 \end{bmatrix}.$$

The minimal and maximal eigenvalues, trace and determinant of the exact solution of the CARE with the above data are $\lambda_n(P) = 0.5081$, $\lambda_1(P) = 0.7767$, $\text{tr}(P) = 1.2848$, and $\det(P) = 0.3946$, respectively. With $\eta = 0.7974$ and $\alpha = 1.5$, the lower matrix bound P_{l1} for the solution P of the CARE (1.1) is found by Theorem 2.1 to be

$$P_{l1} = \begin{bmatrix} 0.3659 & 0.0484 \\ 0.0484 & 0.2427 \end{bmatrix}.$$

With $\alpha = 2$, the lower matrix bound P_{l2} is found by Theorem 2.2 to be

$$P_{l2} = \begin{bmatrix} 0.4292 & 0.0627 \\ 0.0627 & 0.3062 \end{bmatrix}.$$

With $\beta = 3.5$, the lower matrix bound P_{l3} is found by Theorem 2.3 to be

$$P_{l3} = \begin{bmatrix} 0.4739 & 0.0777 \\ 0.0777 & 0.3479 \end{bmatrix}.$$

Using two iterations of Algorithm 3, we can obtain the following tighter lower matrix bounds for the solution of the CARE (1.1):

$$N_1 = \begin{bmatrix} 0.5141 & 0.0718 \\ 0.0718 & 0.3626 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 0.5202 & 0.0707 \\ 0.0707 & 0.3656 \end{bmatrix}.$$

The lower bound derived in [9] gives

$$P \geq \begin{bmatrix} 0.1651 & 0 \\ 0 & 0.1651 \end{bmatrix}.$$

The lower bound proposed in [13] gives

$$P \geq \begin{bmatrix} 0.1710 & 0 \\ 0 & 0.1710 \end{bmatrix}.$$

We shall now compare our bounds with the lower bound P_{l4} proposed in [12]. To simplify the calculation of the lower bound P_{l4} , Lee [12] listed some choices of the tuning matrix R_1 . These choices are re-listed in Table 1.

With $R_1 = (\frac{1}{\epsilon}I - BB^T)^{-1}$ and $\epsilon = 0.1$, P has the lower bound P_{l4} given by

$$P \geq \begin{bmatrix} 0.2911 & 0.0322 \\ 0.0322 & 0.4863 \end{bmatrix}.$$

Table 1

Simple choices of R_1 together with the corresponding matrices G of P_{l4}

R_1	G	Range of parameter ϵ
$(\frac{1}{\epsilon}I - BB^T)^{-1}$	$\frac{1}{\sqrt{\epsilon}}$	$0 < \epsilon < \lambda_1^{-1}(BB^T + AQ^{-1}A^T)$
$(\frac{1}{\epsilon}Q - BB^T)^{-1}$	$\frac{1}{\sqrt{\epsilon}}Q$	$0 < \epsilon < \lambda_1^{-1}[(BB^T + AQ^{-1}A^T)Q^{-1}]$
$(\frac{1}{\epsilon}AA^T - BB^T)^{-1}$	$\frac{1}{\sqrt{\epsilon}}AA^T$	$0 < \epsilon < \lambda_1^{-1}[(BB^T + AQ^{-1}A^T)(AA^T)^{-1}]$
$\frac{1}{\epsilon}I$	$(BB^T + \epsilon I)^{1/2}$	$\epsilon > \lambda_1(A^TQ^{-1}A)$
ϵQ	$(BB^T + \frac{1}{\epsilon}Q^{-1})^{1/2}$	$0 < \epsilon < \lambda_1^{-1}(A^TQAQ^{-1})$
ϵQ^{-1}	$(BB^T + \frac{1}{\epsilon}Q)^{1/2}$	$0 < \epsilon < \lambda_1^{-1}(A^TQ^{-1}AQ^{-1})$
$\epsilon(AA^T)^{-1}$	$(BB^T + \frac{1}{\epsilon}AA^T)^{1/2}$	$0 < \epsilon < \lambda_n(Q)$
$\epsilon(AQA^T)^{-1}$	$(BB^T + \frac{1}{\epsilon}AQA^T)^{1/2}$	$0 < \epsilon < \lambda_n^2(Q)$

With $R_1 = (\frac{1}{\epsilon}Q - BB^T)^{-1}$ and $\epsilon = 0.1$, P has the lower bound P_{l4} given by

$$P \geq \begin{bmatrix} 0.1964 & 0.0100 \\ 0.0100 & 0.1849 \end{bmatrix}.$$

With $R_1 = (\frac{1}{\epsilon}AA^T - BB^T)^{-1}$ and $\epsilon = 0.02$, P has the lower bound P_{l4} given by

$$P \geq \begin{bmatrix} 0.6473 & -0.0454 \\ -0.0454 & 0.0367 \end{bmatrix}.$$

With $R_1 = \frac{1}{\epsilon}I$ and $\epsilon = 4$, P has the lower bound P_{l4} given by

$$P \geq \begin{bmatrix} 0.3419 & -0.0255 \\ -0.0255 & 0.5436 \end{bmatrix}.$$

With $R_1 = \epsilon Q$ and $\epsilon = 0.1$, P has the lower bound P_{l4} given by

$$P \geq \begin{bmatrix} 0.2031 & 0.1709 \\ 0.1709 & 0.4620 \end{bmatrix}.$$

With $R_1 = \epsilon Q^{-1}$ and $\epsilon = 0.5$, P has the lower bound P_{l4} given by

$$P \geq \begin{bmatrix} 0.3348 & 0.0771 \\ 0.0771 & 0.5661 \end{bmatrix}.$$

With $R_1 = \epsilon(AA^T)^{-1}$ and $\epsilon = 0.5$, P has the lower bound P_{l4} given by

$$P \geq \begin{bmatrix} 0.3358 & -0.0186 \\ -0.0186 & 0.4164 \end{bmatrix}.$$

With $R_1 = \epsilon(AQA^T)^{-1}$ and $\epsilon = 0.5$, P has the lower bound P_{l4} given by

$$P \geq \begin{bmatrix} 0.3342 & -0.0075 \\ -0.0075 & 0.2680 \end{bmatrix}.$$

Viewing the above numerical experiments, it can be seen that our bounds are tighter than some of the existing results for some cases. For these examples, it is seen that the bound P_{l3} is the tightest. Additionally, P_{l3} seems to be the least difficult bound to compute.

4. Conclusions

In this paper, new lower matrix bounds for the solution of the CARE have been proposed. Following each bound derivation, an iterative algorithm was given to obtain tighter solution bounds derivation. The advantage of these lower matrix bounds over existing bounds is that they are always calculated when the solution of the CARE exists, and always yield nontrivial bounds even when Q is positive semi-definite. For the first numerical example in which Q is singular, our results yield nontrivial lower matrix bounds for the CARE, whereas the only other existing bound for this case [9] yields the trivial lower bound $P \geq 0$. For the second example when Q is nonsingular, we find that our bounds are tighter than some of the existing results for some cases. The tightness of our bounds and existing bounds depend of the selection of the tuning parameters or tuning matrices involved.

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